The well-posedness for the Navier-Stokes equations and its related equations in the maximal regularity class

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The fundamental equation describing the motion of incompressible viscous fluids for the velocity $u = u(x,t) : \mathbb{R}^N \times (0,T) \to \mathbb{R}^N$ and the pressure $p = p(x,t) : \mathbb{R}^N \times (0,T) \to \mathbb{R}$ is the Navier-Stokes equation :

$$\begin{cases} \partial_t u - \mu\Delta u + (u\cdot\nabla)u + \nabla p = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

where $\partial_t = \partial/\partial t$ and $\mu > 0$ is the viscosity constant. Note that the Navier-Stokes equation is a semilinear equation with a nonlinear term $(u\cdot\nabla)u$. In Kato [3], the unique existence of strong solutions of (1) is proved by the L_p-L_q decay estimates of the Stokes semigroup.

On the other hand, quasilinear equations also appear in fluid dynamics. For instance, free boundary problems for the Navier-Stokes equations and the compressible Navier-Stokes equations. More precisely, the equations in the time-dependent domain are transformed into the equations in a fixed domain by some change of variables to solve the free boundary problems; that problem in the fixed domain is a quasilinear system. It is difficult to solve quasilinear equations by semigroup theory only because of regularity-loss. One of the methods to solve quasilinear equations is the maximal regularity for the linearized problem.

In this lecture, I introduce one of the linear theories to obtain the maximal regularity and how to prove the well-posedness by the maximal regularity estimates, especially the global well-posedness for small initial data in the whole space. Applying these theories, we consider the global well-posedness for a model of nematic liquid crystals in $\mathbb{R}^3$.

1 Maximal Regularity

We introduce the maximal regularity, which is a key tool to solve quasilinear parabolic or parabolic-hyperbolic equations by the Banach fixed point argument. Note that the following approach can be applied to problems with inhomogeneous boundary condition. In this lecture, we mainly discuss the problem in the whole space.

Let $\mathcal{A}$ be the generator of an analytic semigroup on Banach space X. Let us consider the Cauchy problem

$$\begin{cases} \partial_t u - \mathcal{A}u = f & \text{in } \mathbb{R}^N \text{ for } t \in (0,T), \\ u|_{t=0} = u_0 & \text{in } \mathbb{R}^N \end{cases} \tag{2}$$

with given $f \in L_p((0,T), X)$.

Definition 1. $\mathcal{A}$ has the maximal regularity if the Cauchy problem (2) has a unique solution satisfying

$$\|\partial_t u\|_{L_p((0,T),X)} + \|\mathcal{A}u\|_{L_p((0,T),X)} \leq C\|f\|_{L_p((0,T),X)}.$$

The operator-valued Fourier multiplier theorem [5] helps obtain the maximal regularity. To apply this theorem, we prove $\mathcal{R}$-boundedness of solution operator families for the resolvent problem corresponding to (2):

$$\lambda u - \mathcal{A}u = f \quad \text{in } \mathbb{R}^N, \tag{3}$$

where λ is the resolvent parameter varying in a sector

$$\Sigma_{\epsilon,\lambda_0} = \{\lambda \in \mathbb{C} \setminus \{0\} \mid |\arg \lambda| < \pi - \epsilon, |\lambda| \geq \lambda_0\}$$

for $0 < \epsilon < \pi/2$ and $\lambda_0 \geq 0$. Here, we introduce the definition of $\mathcal{R}$-boundedness of operator families and the operator-valued Fourier multiplier theorem.

Definition 2. A family of operators $\mathcal{T} \subset \mathcal{L}(X,Y)$ is called $\mathcal{R}$-bounded on $\mathcal{L}(X,Y)$, if there exist constants $C > 0$ and $p \in [1,\infty)$ such that for any $n \in \mathbb{N}$, $\{T_j\}_{j=1}^n \subset \mathcal{T}$, $\{f_j\}_{j=1}^n \subset X$ and sequences $\{r_j\}_{j=1}^n$ of independent, symmetric, $\{-1,1\}$-valued random variables on $[0,1]$, we have the inequality:

$$\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u)T_j f_j\|_Y^p \, du\Big\}^{1/p} \leq C\Big\{\int_0^1 \|\sum_{j=1}^n r_j(u) f_j\|_X^p \, du\Big\}^{1/p}.$$

The smallest such C is called $\mathcal{R}$-bound of $\mathcal{T}$, which is denoted by $\mathcal{R}_{\mathcal{L}(X,Y)}(\mathcal{T})$.

Remark 3. Definition 2 with $n = 1$ implies the uniform boundedness of the operator family $\mathcal{T}$; therefore, once we obtain the $\mathcal{R}$-bounded solution operator families for (3), a solution u of (3) satisfies the resolvent estimate for any $\lambda \in \Sigma_{\epsilon,\lambda_0}$ and then the linear operator $\mathcal{A}$ generates a analytic semigroup $\{e^{\mathcal{A}t}\}_{t\geq 0}$.

Let $\mathcal{S}(\mathbb{R}, X)$ be the Schwartz space of rapidly decreasing X valued functions.

Definition 4. A Banach space X is said to be a UMD Banach space, if the Hilbert transform is bounded on $L_p(\mathbb{R}, X)$ for some $p \in (1,\infty)$. Here, the Hilbert transform H operating on $f \in \mathcal{S}(\mathbb{R}, X)$ is defined by

$$[Hf](t) = \frac{1}{\pi} \lim_{\epsilon \to 0} \int_{|t-s|>\epsilon} \frac{f(s)}{t-s} \, ds \quad (t \in \mathbb{R}).$$

Remark 5. Let Ω be a domain in $\mathbb{R}^N$. $L_p(\Omega)$ is a UMD Banach space if $1 < p < \infty$.

Theorem 6 (Weis [5]). *Let X and Y be two UMD Banach spaces and $1 < p < \infty$. Let m be a function in $C^1(\mathbb{R} \setminus \{0\}, \mathcal{L}(X,Y))$ such that*

$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{m(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) \leq \kappa_0 < \infty,$$
$$\mathcal{R}_{\mathcal{L}(X,Y)}(\{\tau m'(\tau) \mid \tau \in \mathbb{R} \setminus \{0\}\}) \leq \kappa_1 < \infty$$

with some constant κ_0 and κ_1. Then, the operator

$$T_m f = \mathcal{F}^{-1}[m(\tau)\mathcal{F}[f]],$$

satisfies

$$\|T_m f\|_{L_p(\mathbb{R},Y)} \leq C_p(\kappa_0 + \kappa_1)\|f\|_{L_p(\mathbb{R},X)}$$

for all $f \in \mathcal{S}(\mathbb{R}, X)$ with some positive constant C_p depending on p.

Remark 7. Since $\mathcal{S}(\mathbb{R}, X)$ is dense in $L_p(\mathbb{R}, X)$, T_m is extended to a bounded linear operator on $L_p(\mathbb{R}, X)$. Denoting this extension by T_m again, the above estimate holds for $f \in L_p(\mathbb{R}, X)$.

The following lemma proved by [2, Theorem 3.3] is a sufficient condition for the $\mathcal{R}$-boundedness in $\mathbb{R}^N$.

Lemma 8. *Let $1 < q < \infty$ and let Λ be a subset of $\mathbb{C}$. Let $m(\xi, \lambda)$ be a function defined on $(\mathbb{R}^N \setminus \{0\}) \times \Lambda$ which is infinitely differentiable with respect to $\xi \in \mathbb{R}^N \setminus \{0\}$ for each $\lambda \in \Lambda$. Assume that for any multi-index $\alpha \in \mathbb{N}_0^N$ there exists a positive constant C_α depending on α such that*

$$|\partial_\xi^\alpha m(\xi, \lambda)| \leq C_\alpha |\xi|^{-|\alpha|}$$

for any $(\xi, \lambda) \in (\mathbb{R}^N \setminus \{0\}) \times \Lambda$. Let $M(\lambda)$ be operators defined by

$$[M(\lambda)f](x) = \mathcal{F}^{-1}[m(\xi, \lambda)\mathcal{F}[f](\xi)](x).$$

Then, the operator family $\{M(\lambda) \mid \lambda \in \Lambda\}$ is $\mathcal{R}$-bounded on $\mathcal{L}(L_q(\mathbb{R}^N))$ and

$$\mathcal{R}_{\mathcal{L}(L_q(\mathbb{R}^N))}(\{M(\lambda) \mid \lambda \in \Lambda\}) \leq C_{q,N} \max_{|\alpha| \leq N+2} C_\alpha$$

with some constant $C_{q,N}$ depending on q and N.

2 Model of Nematic Liquid Crystals

The molecules of nematic liquid crystal flows as in a liquid phase; however, they have the orientation order. In order to analyze the biaxial nematic liquid crystal flows, Beris and Edwards [1] proposed the symmetric, traceless matrix as the director fields, which is called Q-tensor. We consider the coupled system by the Navier-Stokes equations with a parabolic-type equation describing the evolution of the director fields Q.

$$\begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla p = \Delta u + \operatorname{Div}(\tau(Q) + \sigma(Q)) & \text{in } \mathbb{R}^3 \text{ for } t \in (0, T), \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}^3 \text{ for } t \in (0, T), \\ \partial_t Q + (u \cdot \nabla)Q - S(\nabla u, Q) = H & \text{in } \mathbb{R}^3 \text{ for } t \in (0, T), \\ (u, Q)|_{t=0} = (u_0, Q_0) & \text{in } \mathbb{R}^3. \end{cases} \tag{4}$$

Here, $u = u(x, t)$ is the fluid velocity and $p = p(x, t)$ is the pressure. For 3×3 matrix field A with $(j, k)^{\text{th}}$ components A_{jk}, the quantity $\operatorname{Div} A$ is a vector with j^{th} component

$\sum_{k=1}^N \partial_k A_{jk}$, where $\partial_k = \partial/\partial x_k$. The tensors $S(\nabla u, Q)$, $\tau(Q)$, and $\sigma(Q)$ are

$$
\begin{aligned}
S(\nabla u, Q) &= (\xi D(u) + W(u))\left(Q + \frac{I}{3}\right) \\
&\qquad + \left(Q + \frac{I}{3}\right)(\xi D(u) - W(u)) - 2\xi\left(Q + \frac{I}{3}\right)Q : \nabla u, \\
\tau(Q) &= 2\xi H : Q\left(Q + \frac{I}{3}\right) - \xi\left[H\left(Q + \frac{I}{3}\right) + \left(Q + \frac{I}{3}\right)H\right] - \nabla Q \odot \nabla Q, \\
\sigma(Q) &= QH - HQ,
\end{aligned}
$$

where $D(u) = (\nabla u + (\nabla u)^T)/2$ and $W(u) = (\nabla u - (\nabla u)^T)/2$ denote the symmetric and antisymmetric part of ∇u, respectively. A scalar parameter $\xi \in \mathbb{R}$ denotes the ratio between the tumbling and the aligning effects that a shear flow would exert over the directors. Furthermore, I is the 3×3 identity matrix,

$$
H = \Delta Q - aQ + b(Q^2 - \mathrm{tr}(Q^2)I/3) - c\,\mathrm{tr}(Q^2)Q,
$$

and the (i, j) component of $\nabla Q \odot \nabla Q$ is $\sum_{\alpha,\beta=1}^3 \partial_i Q_{\alpha\beta}\partial_j Q_{\alpha\beta}$.

We consider the global well-posedness for (4) for small initial data in the following solution class:

$$
\begin{aligned}
u &\in H_p^1((0, T), L_q(\mathbb{R}^3)^3) \cap L_p((0, T), H_q^2(\mathbb{R}^3)^3), \\
Q &\in H_p^1((0, T), H_q^1(\mathbb{R}^3; S_0)) \cap L_p((0, T), H_q^3(\mathbb{R}^3; S_0))
\end{aligned}
$$

with certain p and q, where $S_0 = \{Q : 3 \times 3 \text{ matrix} \mid Q = Q^T, \mathrm{tr}Q = 0\}$. This result is based on [4].

References

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